

# (CO-)HOMOLOGY OF REES SEMIGROUP ALGEBRAS

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ABSTRACT. Let  $S$  be a Rees semigroup, and let  $\ell^1(S)$  be its convolution semigroup algebra. Using Morita equivalence we show that bounded Hochschild homology and cohomology of  $\ell^1(S)$  is isomorphic to those of the underlying discrete group algebra.

## 1. INTRODUCTION

In this paper we calculate the simplicial cohomology of the  $\ell^1$ -algebra of Rees semigroups, motivated by the explicit computations of the first [1] and second order cohomology groups [3] of several Banach algebras. These papers contain a number of results showing that the simplicial cohomology (in dimensions 1 and 2 respectively) of  $\ell^1$ -semigroup algebras are trivial for many of the fundamental examples of semigroups. The first paper considers the first order simplicial and cyclic cohomology of Rees semigroup algebras, the bicyclic semigroup algebra and the free semigroup algebra. The second paper [3] shows that the second simplicial cohomology vanishes for the semigroups  $\mathbb{Z}_+$ , semilattice semigroups and Clifford semigroups. It should be noted that in each of these papers the arguments are mostly ad hoc. These two papers are followed by papers involving some of the current authors [4] and [8] which cover the case of the third cohomology groups. In these papers there is some attempt at systematic methods which might be adapted to cover the case of all higher cohomology groups, but the authors were not able to do this at that time. Finally, later papers [5], [6], [7] show how to calculate the higher order simplicial cohomology groups of some of these algebras. These later papers do not use ad hoc calculations, but general homological machinery, such as the Connes-Tsygan long exact sequence and topological simplicial homology to deduce their results. The present paper belongs to the latter family of papers, in that it uses general homological tools.

The proofs in this paper are based firmly on the Morita equivalence methods developed by the second named author [9] and [10] which apply to a general Banach algebra context. We

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shall utilize that for the class of so-called self-induced Banach algebras, bounded Hochschild (co-)homology with appropriate coefficients is Morita invariant.

We briefly describe our general approach. For a semigroup,  $T$ , with an absorbing zero,  $\emptyset$ , there are two Banach algebras that naturally arise, the discrete convolution algebra  $\ell^1(T)$ , and the reduced algebra  $\mathcal{A}(T)$  (to be defined below) associated with the inclusion  $\emptyset \hookrightarrow T$ . In the case of a Rees semigroup,  $S$ , with underlying group  $G$ , we prove that  $\mathcal{A}(S)$  is Morita equivalent to  $\ell^1(G)$ . This may be interpreted as a manifestation of a basic fact from algebraic topology that for a path connected space  $X$  the fundamental groupoid  $\pi(X)$  and the based homotopy group  $\pi_1(X, a)$  are equivalent categories.

Our results then hinge on Theorem 3.1 in which a natural isomorphism between the (co-)homology of  $\mathcal{A}(S)$  and of the underlying group algebra is established by means of Morita equivalence. Using an excision result from [15] we prove that the (co-)homology of  $\ell^1(S)$  and of  $\mathcal{A}(S)$  are isomorphic for a large class of coefficient modules. Since  $\ell^1(S)$  and  $\mathcal{A}(S)$  both are H-unital (cf. [17]), we further obtain (co-)homology results for the forced unitizations  $\ell^1(S)^\sharp$  and  $\mathcal{A}(S)^\sharp$ . From this we derive our main result that for a Rees semigroup,  $S$ , the simplicial cohomology  $\mathcal{H}^n(\ell^1(S), \ell^1(S)^*)$  ( $n \geq 1$ ) of the semigroup algebra is isomorphic to the simplicial cohomology  $\mathcal{H}^n(\ell^1(G), \ell^1(G)^*)$  ( $n \geq 1$ ) of the underlying group algebra.

## 2. BASICS

A *completely 0-simple semigroup* is a semigroup which has a 0, has no proper ideal other than  $\{0\}$ , and has a primitive idempotent, that is a minimal idempotent  $e$  in the set of non-zero idempotents (or, precisely, an idempotent  $e \neq 0$  such that if  $ef = fe = f \neq 0$  for an idempotent  $f$ , then  $e = f$ ). Such semigroups arise naturally and notably in classification theory as quotients of ideals. It is an important result that any completely 0-simple semigroup is isomorphic to what we call a Rees semigroup (and conversely). For this result and more background, see [12, Chapter 3 and Theorem 3.2.3]. We now give the definition of a Rees semigroup and some special cases of Rees semigroups to illustrate how many natural semigroups are of this form.

The data which are required to define a Rees semigroup are given by two index sets  $I$  and  $\Lambda$ , a group  $G$  and a *sandwich matrix*  $P$ . We introduce two zero elements: the first,  $\mathbf{o}$ , is an absorbing element adjoined to  $G$  to make the semigroup denoted  $G^\mathbf{o}$ . The second element,  $\emptyset$ , is an absorbing zero for the Rees semigroup itself. The sandwich matrix  $P = (p_{\lambda i})$ , is a set of elements of  $G^\mathbf{o}$  indexed by  $\Lambda \times I$ , such that each row and column of  $P$  has at least one

non-zero entry. The Rees semigroup  $S$  is then the set  $I \times G \times \Lambda \cup \{\emptyset\}$ , where  $\emptyset$  is an absorbing zero for the semigroup, and the other products are defined by the rule

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & (p_{\lambda j} \neq \mathbf{o}) \\ \emptyset & (p_{\lambda j} = \mathbf{o}). \end{cases}$$

**2.1. Examples.** There are two extreme, degenerate cases which provide good intuition for the logic of calculations.

The first is the case where  $I$  and  $\Lambda$  are just singletons and the sandwich matrix consists of the identity of  $G$ . In this case  $S$  is just  $G^\circ$ . The reduced semigroup algebra,  $\mathcal{A}(S)$  defined below, will give us  $\ell^1(G)$  in this case.

The second case has the group,  $G$ , being trivial and the index sets being both equal to the set of the first  $n$  natural numbers  $\{1, 2, \dots, n\}$ . The sandwich matrix is diagonal with the group identity repeated along the diagonal. If we identify  $G$  with  $\{1\} \subseteq \mathbb{C}$  the Rees semigroup becomes the system of matrix units together with the zero-matrix,  $S = \{E_{ij} : 1 \leq i, j \leq n\} \cup \{0\}$ , and  $\ell^1(S)$  is the algebra of complex 2-block  $(n+1) \times (n+1)$  matrices consisting of an upper  $n \times n$  block and a lower  $1 \times 1$  block. The reduced semigroup algebra,  $\mathcal{A}(S)$ , does not have this deficiency and is exactly the matrix algebra,  $M_n(\mathbb{C})$ .

Our third example is from homotopy theory and is almost generic for the concept of Rees semigroups. Recall that a small category in which every morphism is invertible is called a *groupoid*. It is *connected* if there is a morphism between each pairs of objects. The canonical example is  $\pi(X)$  the *fundamental groupoid* of a path connected topological space  $X$ . The objects are the points of  $X$  and for  $x, y \in X$  the morphism set  $\pi(X)(x, y)$  is the set of homotopy classes relative to  $x, y$  of paths from  $x$  to  $y$  with composition derived from products of paths. In particular  $\pi(X)(x, x) = \pi_1(X, x)$  for each  $x \in X$ , so that the fundamental group of  $X$  at  $x$  is identified with the full subcategory of  $\pi(X)$  with just one object  $x$ . Since  $X$  is path connected  $\pi_1(X, x) \cong \pi_1(X, y)$  for each  $x, y \in X$ , for details see [2]

Fix  $a \in X$  and set  $G = \pi_1(X, a)$ . We associate Rees semigroups to  $\pi(X)$  in the following way. For each  $x \in X$  choose  $s_x \in \pi(X)(a, x)$  and  $t_x \in \pi(X)(x, a)$ . This gives bijections  $\psi_{x,y} : \pi(X)(x, y) \rightarrow G$

$$\psi_{x,y}(\gamma) = s_x \gamma t_y, \quad x, y \in X, \gamma \in \pi(X)(x, y).$$

Then we get

$$\psi_{x,z}(\gamma\gamma') = \psi_{x,y}(\gamma)(s_y t_y)^{-1} \psi_{y,z}(\gamma'), \quad x, y, z \in X, \gamma \in \pi(X)(x, y), \gamma' \in \pi(X)(y, z).$$

Let  $I, \Lambda$  be sets and consider maps  $\alpha: I \rightarrow X$ ,  $\beta: \Lambda \rightarrow X$ . We define multiplication on  $(I \times G \times \Lambda) \cup \emptyset$  as follows. For  $i, j \in I$ ,  $g, h \in G$ ,  $\lambda, \mu \in \Lambda$ , put  $\gamma = \psi_{\alpha(i), \beta(\lambda)}^{-1}(g)$ ,  $\gamma' = \psi_{\alpha(j), \beta(\mu)}^{-1}(h)$  and set

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, \psi_{\alpha(i), \beta(\mu)}(\gamma\gamma'), \mu) & (\alpha(j) = \beta(\lambda)) \\ \emptyset & (\alpha(j) \neq \beta(\lambda)) \end{cases}$$

so that the product is simply given by products of paths, when defined. The sandwich matrix is

$$p_{\lambda i} = \begin{cases} (s_{\alpha(i)} t_{\beta(\lambda)})^{-1} & (\alpha(i) = \beta(\lambda)) \\ \mathbf{o} & (\alpha(i) \neq \beta(\lambda)). \end{cases}$$

The condition that the sandwich matrix  $(p_{\lambda i})$  has a non-zero entry in each row and each column is  $\alpha(I) = \beta(\Lambda)$ .

It is a simple fact, but crucial to the use of groupoids in algebraic topology cf. [2, Chp. 8], that the fundamental groupoid  $\pi(X)$  and the based homotopy group  $\pi_1(X, a)$  are equivalent categories. Our main result on Morita equivalence is a manifestation of this fact in the setting of convolution Banach algebras.

**2.2. Background on homological algebra.** The main result of this paper concerns bounded (co-)homology of Banach algebras, so we will give a brief description of the theory of (co-)homology of Banach algebras, as we fix the notation we will use for this paper. For further details we refer to [11].

For a Banach algebra  $\mathcal{A}$  we denote the categories of left (right) Banach  $\mathcal{A}$ -modules and bounded module homomorphisms by  $\mathcal{A}\text{-}\mathbf{mod}$  (respectively  $\mathbf{mod}\text{-}\mathcal{A}$ ). If  $\mathcal{B}$  is also a Banach algebra, the category of Banach  $\mathcal{A}\text{-}\mathcal{B}$  bimodules and bounded homomorphisms is  $\mathcal{A}\text{-}\mathbf{mod}\text{-}\mathcal{B}$ . A *full subcategory* is a subcategory  $\mathfrak{C}$  which includes all morphisms between objects of  $\mathfrak{C}$ .

Let  $\mathcal{A}$  be a Banach algebra,  $\mathcal{A}^\#$  be its forced unitization and  $X \in \mathcal{A}\text{-}\mathbf{mod}$ . The *bar resolution* of  $X$  is the complex

$$\mathbb{B}(\mathcal{A}, X): 0 \leftarrow X \leftarrow \mathcal{A}^\# \hat{\otimes} X \leftarrow \mathcal{A}^\# \hat{\otimes} \mathcal{A} \hat{\otimes} X \leftarrow \cdots \leftarrow \mathcal{A}^\# \hat{\otimes} \mathcal{A} \hat{\otimes} \cdots \hat{\otimes} \mathcal{A} \hat{\otimes} X \leftarrow \cdots$$

with boundary maps

$$b(a_1 \otimes \cdots \otimes a_n \otimes x) = \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes x + (-1)^{n-1} a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n x$$

Similarly we define the bar resolution for  $X \in \mathbf{mod}\text{-}\mathcal{A}$ . It is a standard fact that  $\mathbb{B}(\mathcal{A}, X)$  is contractible.

The *simplicial complex* of  $\mathcal{A}$  is the subcomplex of  $\mathbb{B}(\mathcal{A}, \mathcal{A})$  with the first tensor factor in  $\mathcal{A}$  rather than  $\mathcal{A}^\#$ :

$$0 \leftarrow \mathcal{A} \leftarrow \mathcal{A} \widehat{\otimes} \mathcal{A} \leftarrow \cdots \leftarrow \mathcal{A}^{n\widehat{\otimes}} \leftarrow \cdots \leftarrow$$

If the dual complex of the simplicial complex

$$0 \rightarrow \mathcal{A}^* \rightarrow (\mathcal{A} \widehat{\otimes} \mathcal{A})^* \rightarrow \cdots \rightarrow (\mathcal{A}^{n\widehat{\otimes}})^* \rightarrow \cdots \rightarrow$$

splits as a complex of Banach spaces, then  $\mathcal{A}$  is *H-unital* [17].

A module  $X \in \mathcal{A}\text{-mod-}\mathcal{B}$  is *induced* if the multiplication

$$\mathcal{A} \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{B}} \mathcal{B} \rightarrow X : a \otimes_{\mathcal{A}} x \otimes_{\mathcal{B}} b \mapsto axb$$

is an isomorphism. If  $\mathcal{A}$  is induced as a module in  $\mathcal{A}\text{-mod-}\mathcal{A}$  then  $\mathcal{A}$  is *self-induced*.

A bounded linear map  $L: E \rightarrow F$  between Banach spaces is *admissible* if  $\ker E$  and  $\operatorname{im} E$  are complemented as Banach spaces in  $E$  respectively  $F$ .

A module  $P \in \mathcal{A}\text{-mod}$  is *(left) projective* if, for every admissible epimorphism  $q: Y \rightarrow Z$ , all lifting problems in  $\mathcal{A}\text{-mod}$

$$\begin{array}{ccc} & P & \\ \swarrow \gamma & \downarrow \phi & \\ Y & \xrightarrow{q} & Z \longrightarrow 0 \end{array}$$

can be solved. If all, not just admissible, lifting problems can be solved, then  $P$  is *strictly projective*. The module  $P$  is *(left) flat* if for every admissible short exact sequence in  $\mathbf{mod}\text{-}\mathcal{A}$

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

the sequence

$$0 \rightarrow X \widehat{\otimes}_{\mathcal{A}} P \rightarrow Y \widehat{\otimes}_{\mathcal{A}} P \rightarrow Z \widehat{\otimes}_{\mathcal{A}} P \rightarrow 0$$

is exact, and *strictly flat* if the requirement of admissibility can be omitted.

The fundamental concept of our approach is Morita equivalence.

**Definition 2.1.** Two self-induced Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  are Morita equivalent if there are induced modules  $P \in \mathcal{B}\text{-mod-}\mathcal{A}$  and  $Q \in \mathcal{A}\text{-mod-}\mathcal{B}$  so that

$$P \widehat{\otimes}_{\mathcal{A}} Q \cong \mathcal{B} \quad \text{and} \quad Q \widehat{\otimes}_{\mathcal{B}} P \cong \mathcal{A},$$

where the isomorphisms are implemented by bounded bilinear balanced module maps  $[\cdot, \cdot]: P \times Q \rightarrow \mathcal{B}$  and  $(\cdot, \cdot): Q \times P \rightarrow \mathcal{A}$  satisfying

$$[p, q].p' = p.(q, p') \quad q.[p, q'] = (q, p).q' \quad p, p' \in P; \quad q, q' \in Q.$$

Our objective is to describe bounded Hochschild (co-)homology of Rees semigroup algebras in terms of the (co-)homology of the algebra of the underlying group. First we define homology.

**Definition 2.2.** For  $X \in \mathcal{A}\text{-mod-}\mathcal{A}$  the Hochschild complex is

$$0 \leftarrow X \leftarrow X \widehat{\otimes} \mathcal{A} \leftarrow \cdots \leftarrow X \widehat{\otimes} \mathcal{A}^{n \widehat{\otimes}} \leftarrow \cdots$$

with boundary maps given as

$$\begin{aligned} \delta(x \otimes a_1 \otimes \cdots \otimes a_n) &= xa_1 \otimes a_2 \otimes \cdots \otimes a_n + x \otimes \sum_{k=1}^{n-1} (-1)^k a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

The bounded Hochschild homology of  $\mathcal{A}$  with coefficients in  $X$ ,  $\mathcal{H}_n(A, X)$ ,  $n = 0, 1, \dots$ , is the homology of this complex. The bounded Hochschild cohomology of  $\mathcal{A}$  with coefficients in the dual module  $X^*$ ,  $\mathcal{H}^n(A, X^*)$ ,  $n = 0, 1, \dots$ , is the cohomology of the dual complex

$$0 \rightarrow X^* \rightarrow (\mathcal{A} \widehat{\otimes} X)^* \rightarrow \cdots \rightarrow (\mathcal{A}^{n \widehat{\otimes}} \widehat{\otimes} X)^* \rightarrow \cdots$$

Our main result hinges on the fact that bounded Hochschild homology and cohomology under certain conditions are Morita invariant, cf [10]. We state the version that we need in the paper.

**Theorem 2.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be Morita equivalent Banach algebras with implementing modules  $P \in \mathcal{B}\text{-mod-}\mathcal{A}$  and  $Q \in \mathcal{A}\text{-mod-}\mathcal{B}$ . If  $P$  is right flat as a module in  $\text{mod-}\mathcal{A}$  and left flat as a module in  $\mathcal{B}\text{-mod}$ , then there are natural isomorphisms

$$\mathcal{H}_n(\mathcal{A}, X) \cong \mathcal{H}_n(\mathcal{B}, P \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{A}} Q) \quad \text{and} \quad \mathcal{H}^n(\mathcal{A}, X^*) \cong \mathcal{H}^n(\mathcal{B}, (P \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{A}} Q)^*)$$

for all induced modules  $X \in \mathcal{A}\text{-mod-}\mathcal{A}$ .

*Proof.* To establish the homology statement we briefly recall the Waldhausen first quadrant double complex. For details we refer to [10, pp. 132-133]. On the horizontal axis it is the Hochschild complex in  $\mathcal{A}\text{-mod-}\mathcal{A}$  for  $X$  and on the vertical axis it is the Hochschild complex

in  $\mathcal{B}\text{-mod-}\mathcal{B}$  for  $P\widehat{\otimes}_{\mathcal{A}}X\widehat{\otimes}_{\mathcal{A}}Q$ . For  $n \geq 1$  the  $n$ 'th row is

$$P\widehat{\otimes}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, X\widehat{\otimes}_{\mathcal{A}}Q\widehat{\otimes}\mathcal{B}^{\otimes(n-1)})$$

and the  $n$ 'th column is

$$\mathbb{B}(\mathcal{B}, \mathcal{A}^{\otimes(n-1)}\widehat{\otimes}X\widehat{\otimes}_{\mathcal{A}}Q)\widehat{\otimes}_{\mathcal{B}}P.$$

By [9, Lemma 6.1] it suffices that rows and columns are acyclic for  $n \geq 1$ . As the bar complexes are contractible, this follows from the flatness properties of  $P$ .

Dualizing the Waldhausen double complex we obtain the cohomology statement.  $\square$

**2.3. Semigroup algebras.** Given a semigroup  $T$ , the semigroup algebra is the Banach space  $\ell^1(T)$ , equipped with the product which extends the product defined on the natural basis from  $T$ , by bilinearity, to the whole of  $\ell^1(T)$ . Throughout, we denote an element of the natural basis for  $\ell^1(T)$ , corresponding to  $t \in T$ , by  $t$  itself.

An *absorbing element* for a semigroup  $T$  is an element  $\emptyset \in T$  so that  $t\emptyset = \emptyset t = \emptyset$  for all  $t \in T$ . Obviously there is at most 1 absorbing element in  $T$ . If  $\emptyset$  is absorbing, then  $\mathbb{C}\emptyset$  is a 1-dimensional 2-sided ideal of  $\ell^1(T)$ . Our calculations are more easily done modulo this ideal.

**Definition 2.4.** *Let  $T$  be a semigroup with absorbing element  $\emptyset$ . The reduced semigroup algebra is*

$$\mathcal{A}(T) = \ell^1(T)/\mathbb{C}\emptyset.$$

*As a Banach space,  $\mathcal{A}(T)$  is isometrically isomorphic to  $\ell^1(T \setminus \{\emptyset\})$  and the multiplication is given by*

$$st = \begin{cases} st & \text{if } st \neq \emptyset \text{ in } T, \\ 0 & \text{if } st = \emptyset \text{ in } T. \end{cases}$$

*We note that if the semigroup satisfies  $T^2 = T$  then the multiplication maps*

$$\ell^1(T)\widehat{\otimes}\ell^1(T) \rightarrow \ell^1(T) \text{ and } \mathcal{A}(T)\widehat{\otimes}\mathcal{A}(T) \rightarrow \mathcal{A}(T)$$

*are both surjective.*

*For  $X \in \ell^1(T)\text{-mod-}\ell^1(T)$  the reduced module is*

$$\widetilde{X} = \frac{X}{\emptyset X + X\emptyset}.$$

*The reduced module is canonically a module in  $\mathcal{A}(T)\text{-mod-}\mathcal{A}(T)$ .*

**Example 2.5.** *The concept of reduced semigroup algebra fits in the more general context of extension of semigroups. If  $I$  is a semigroup ideal of a semigroup  $T$ , then the Rees factor semigroup  $T/I$  has the equivalence class  $I$  as an absorbing zero and we get the corresponding admissible extension of Banach algebras*

$$0 \rightarrow \ell^1(I) \rightarrow \ell^1(T) \rightarrow \mathcal{A}(T/I) \rightarrow 0.$$

Note that every module in  $X \in \mathcal{A}(T)\text{-}\mathbf{mod}\text{-}\mathcal{A}(T)$  can be obtained as a reduced module from a module in  $\ell^1(T)\text{-}\mathbf{mod}\text{-}\ell^1(T)$ , simply by extending to an action of  $T$  by  $\emptyset X = X\emptyset = \{0\}$ , so that in this case  $\tilde{X} = X$ .

From [15], we get the following proposition relating the Hochschild homology and cohomology of  $\ell^1(T)$  and  $\mathcal{A}(T)$ .

**Proposition 2.6.** *Let  $X \in \ell^1(T)\text{-}\mathbf{mod}\text{-}\ell^1(T)$  be such  $\emptyset X = X\emptyset$ . Then*

$$\mathcal{H}_n(\ell^1(T), X) \cong \mathcal{H}_n(\mathcal{A}(T), \tilde{X}) \text{ and } \mathcal{H}^n(\ell^1(T), X^*) \cong \mathcal{H}^n(\mathcal{A}(T), \tilde{X}^*)$$

for  $n = 0, 1, \dots$

*Proof.* By [15, Theorem 4.5] we have the long exact sequence

$$\dots \rightarrow \mathcal{H}_n(\mathbb{C}\emptyset, \emptyset X) \rightarrow \mathcal{H}_n(\ell^1(T), X) \rightarrow \mathcal{H}_n(\mathcal{A}(T), \tilde{X}) \rightarrow \mathcal{H}_{n+1}(\mathbb{C}\emptyset, \emptyset X) \rightarrow \dots$$

As  $\mathbb{C}\emptyset \cong \mathbb{C}$  we have  $\mathcal{H}_n(\mathbb{C}\emptyset, \emptyset X) = \{0\}$  for all  $n \geq 0$ , yielding the claim. A similar application of [15, Theorem 4.5] to cohomology gives the other statement.  $\square$

As a consequence, since our concern is to determine Hochschild homology and cohomology, we shall work with reduced semigroup algebras in the following.

**2.4. The Rees semigroup algebra.** For a Rees semigroup with index sets  $I$  and  $\Lambda$  over a group  $G$ , set

$${}_i S_\lambda = \{i\} \times G \times \{\lambda\} \quad i \in I, \lambda \in \Lambda.$$

Then

$$\mathcal{A}(S) = \bigoplus_{\substack{i \in I \\ \lambda \in \Lambda}} \ell^1({}_i S_\lambda)$$

is a decomposition of  $\mathcal{A}(S)$  into an  $\ell^1$ -direct sum of subalgebras such that

$$\ell^1({}_i S_\lambda) \cong \ell^1(G), \text{ if } p_{\lambda i} \neq \mathbf{o}$$

$$\ell^1({}_i S_\lambda) \cdot \ell^1({}_j S_\mu) = \{0\}, \text{ if } p_{\lambda j} = \mathbf{o}.$$

The isomorphism above is implemented by the semigroup isomorphism

$$G \rightarrow {}_i S_\lambda : g \mapsto (i, gp_{\lambda i}^{-1}, \lambda).$$

Since  $\ell^1({}_i S_\lambda) \cdot \ell^1({}_j S_\mu) \subseteq \ell^1({}_i S_\mu)$  this decomposition is organized as a rectangular band. We further put

$$\begin{aligned} {}_i S &= \{i\} \times G \times \Lambda, \quad i \in I \\ S_\lambda &= I \times G \times \{\lambda\}, \quad \lambda \in \Lambda \end{aligned}$$

so that  $\ell^1({}_i S) = \bigoplus_{\lambda \in \Lambda} \ell^1({}_i S_\lambda)$  for each  $i \in I$  is a closed right ideal of  $\mathcal{A}(S)$  and  $\ell^1(S_\lambda) = \bigoplus_{i \in I} \ell^1({}_i S_\lambda)$  for each  $\lambda \in \Lambda$  is a closed left ideal of  $\mathcal{A}(S)$ .

For the remainder of this section we fix a Rees semigroup and establish a number of generic properties.

**Proposition 2.7.** *Each  $\ell^1({}_i S)$ ,  $i \in I$ , is a left unital right ideal and each  $\ell^1(S_\lambda)$ ,  $\lambda \in \Lambda$ , is a right unital left ideal.*

*Proof.* Given an element of the indexing set  $i \in I$  there is, by the property of the sandwich matrix  $P$ , a nonzero entry  $p_{\mu i}$ , for some  $\mu \in \Lambda$ , (as each row and column has a non-zero entry). We define  $e_i = (i, p_{\mu i}^{-1}, \mu)$ . The element  $e_i$  acts as a left identity for  $s \in {}_i S$  as

$$e_i s = (i, p_{\mu i}^{-1}, \mu)(i, g, \lambda) = (i, p_{\mu i}^{-1} p_{\mu i} g, \lambda) = (i, g, \lambda) = s.$$

In particular  $e_i$  is idempotent, and is clearly a left identity for  $\ell^1({}_i S)$ . Similarly on the right we have a non-zero element  $p_{\lambda j}$  of  $P$ , which gives the required element as  $(j, p_{\lambda j}^{-1}, \lambda)$ .  $\square$

Note that the idempotent  $e_i$  is not necessarily unique as we can form such an idempotent using any index from  $\Lambda$  which gives a non-zero entry in  $P$ . However, in what follows it will be useful to have a fixed family of left and right idempotents in mind.

**Definition 2.8.** *We fix a family of left (and right) idempotents denoted  $\{e_i\}_{i \in I}$  (and  $\{f_\lambda\}_{\lambda \in \Lambda}$ ), which are left (respectively right) units for  $\ell^1({}_i S)$  (respectively  $\ell^1(S_\lambda)$ ).*

**Proposition 2.9.** *For each  $i \in I$ , the right ideal  $\ell^1({}_i S)$  is strictly projective in  $\mathbf{mod}\text{-}\mathcal{A}(S)$  and, for each  $\lambda \in \Lambda$ , the left ideal  $\ell^1(S_\lambda)$  is strictly projective in  $\mathcal{A}(S)\text{-}\mathbf{mod}$ .*

*Proof.* We give the proof for  $\ell^1({}_i S)$  as the other is completely analogous. Let  $q: Y \rightarrow Z$  be an epimorphism in  $\mathbf{mod}\text{-}\mathcal{A}(S)$  and consider the lifting problem

$$\begin{array}{ccc} & \ell^1({}_i S) & \\ & \downarrow \phi & \\ Y & \xrightarrow{q} & Z \longrightarrow 0 \end{array}$$

Choose  $y_i \in Y$  so that  $q(y_i) = \phi(e_i)$  and define for  $s \in I \times G \times \Lambda$

$$\tilde{\phi}(e_i s) = y_i e_i s.$$

As  ${}_i S = e_i(I \times G \times \Lambda)$  in  $\mathcal{A}(S)$  the universal property of  $\ell^1$ -spaces provides a bounded linear map  $\tilde{\phi}: \ell^1({}_i S) \rightarrow Y$  so that  $q \circ \tilde{\phi} = \phi$ . Clearly  $\tilde{\phi}$  is a right module map.  $\square$

**Corollary 2.10.**  $\mathcal{A}(S)$  is strictly projective in  $\mathbf{mod}\text{-}\mathcal{A}(S)$  and in  $\mathcal{A}(S)\text{-}\mathbf{mod}$ .

*Proof.* We utilize the direct sum decomposition  $\mathcal{A}(S) = \bigoplus_{i \in I} \ell^1({}_i S)$  in  $\mathbf{mod}\text{-}\mathcal{A}(S)$ . Consider the lifting problem

$$\begin{array}{ccc} & \ell^1({}_i S) & \\ & \downarrow \kappa_i & \\ & \mathcal{A}(S) & \\ & \downarrow \phi & \\ Y & \xrightarrow{q} & Z \longrightarrow 0 \end{array}$$

(Note: In the original image, there is a diagonal arrow from  $\ell^1({}_i S)$  to  $Y$  labeled  $\tilde{\phi}_i$ , and a dashed arrow from  $\mathcal{A}(S)$  to  $Y$ .)

where  $\kappa_i$ ,  $i \in I$ , are the natural inclusions and  $\tilde{\phi}_i$ ,  $i \in I$ , are the lifts of  $\phi \circ \kappa_i$  constructed in 2.9. By the open mapping theorem applied to  $q$ , we can choose the elements  $y_i$  such that  $\|y_i\| \leq C$  for some constant  $C$ , and therefore such that  $\|\tilde{\phi}_i\| \leq C$  for all  $i \in I$ . Thus there is a unique module map  $\tilde{\phi}: \mathcal{A}(S) \rightarrow Y$  with  $\tilde{\phi} \circ \kappa_i = \tilde{\phi}_i$  for all  $i \in I$ . Since both  $q \circ \tilde{\phi}$  and  $\phi$  complete the direct sum diagram for the maps  $\phi \circ \kappa_i$ , it follows by uniqueness of universal elements that  $q \circ \tilde{\phi} = \phi$ .

The case of left projectivity is completely analogous.  $\square$

**Corollary 2.11.**  $\mathcal{A}(S)$  and  $\ell^1(S)$  are  $H$ -unital. In particular  $\mathcal{A}(S)$  and  $\ell^1(S)$  are self-induced.

*Proof.* The statement about  $\mathcal{A}(S)$  is a general fact about one-sided projective Banach algebras with surjective multiplication  $\hat{\mathcal{A}} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ . Let  $\rho: \mathcal{A} \rightarrow \hat{\mathcal{A}} \hat{\otimes} \mathcal{A}$  be a splitting of multiplication provided by right projectivity of  $\mathcal{A}$ . Then  $\rho \otimes \mathbf{1}: \mathcal{A}^{\hat{\otimes} n} \rightarrow \mathcal{A}^{\hat{\otimes}(n+1)}$  is a contracting

homotopy of the simplicial complex:

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \mathcal{A} & \xleftarrow{b} & \mathcal{A}^{\widehat{\otimes} 2} & \xleftarrow{b} & \cdots \\
 & & \downarrow 1 & \searrow \rho & \downarrow 1 & \searrow \rho \otimes 1 & \\
 0 & \longleftarrow & \mathcal{A} & \xleftarrow{b} & \mathcal{A}^{\widehat{\otimes} 2} & \xleftarrow{b} & \mathcal{A}^{\widehat{\otimes} 3} \longleftarrow \cdots
 \end{array}$$

For  $a_1 \otimes \cdots \otimes a_n \in \mathcal{A}^{\widehat{\otimes} n}$

$$\begin{aligned}
 \rho \otimes \mathbf{1}(b(a_1 \otimes \cdots \otimes a_n)) &= \rho(a_1 a_2) \otimes a_3 \otimes \cdots \otimes a_n - \rho(a_1) \otimes b(a_2 \otimes \cdots \otimes a_n) \\
 &= \rho(a_1) a_2 \otimes \cdots \otimes a_n - \rho(a_1) \otimes b(a_2 \otimes \cdots \otimes a_n),
 \end{aligned}$$

and

$$\begin{aligned}
 b(\rho \otimes \mathbf{1}(a_1 \otimes \cdots \otimes a_n)) &= a_1 \otimes \cdots \otimes a_n \\
 &\quad - \rho(a_1) a_2 \otimes \cdots \otimes a_n + \rho(a_1) \otimes b(a_2 \otimes \cdots \otimes a_n).
 \end{aligned}$$

An equivalent formulation of H-unitality is that  $\mathcal{H}_n(\mathcal{A}, X) = \{0\}$  for all trivial modules, i.e. modules with  $\mathcal{A}X = X\mathcal{A} = \{0\}$ . Since  $\widetilde{X} = X$  for any trivial module the statements about  $\ell^1(S)$  follow from 2.6  $\square$

**2.5. Morita equivalence.** Now fix one of the idempotents  $e = (i, p_{\lambda i}^{-1}, \lambda)$  and put

$$P = e\mathcal{A}(S) \text{ and } Q = \mathcal{A}(S)e,$$

so that  $P$  is the closed right ideal  $\ell^1(iS)$  and  $Q$  is the closed left ideal  $\ell^1(S_\lambda)$ . Let further

$$\mathcal{B} = e\mathcal{A}(S)e,$$

so that  $\mathcal{B} = \ell^1(iS_\lambda) \cong \ell^1(G)$ .

For brevity in this section, let  $\mathcal{A} = \mathcal{A}(S)$ . Then  $P \in \mathcal{B}\text{-mod-}\mathcal{A}$  and  $Q \in \mathcal{A}\text{-mod-}\mathcal{B}$ . Our main result is

**Theorem 2.12.** *The modules  $P$  and  $Q$  give a Morita equivalence of  $\mathcal{A}$  and  $\mathcal{B}$ , that is, multiplication gives bimodule isomorphisms*

$$\mathcal{A} \cong Q \widehat{\otimes}_{\mathcal{B}} P \text{ and } \mathcal{B} \cong P \widehat{\otimes}_{\mathcal{A}} Q.$$

*Proof.* Clearly the multiplication  $P \widehat{\otimes}_{\mathcal{B}} Q \rightarrow \mathcal{B}$  is surjective, see 2.4. Now suppose that

$$\sum_n e a_n b_n e = 0 \text{ for } \sum_n \|e a_n\| \|b_n e\| < \infty, \ a_n, b_n \in \mathcal{A}.$$

Then

$$\sum_n ea_n \otimes_{\mathcal{A}} b_n e = \sum_n ea_n b_n e \otimes_{\mathcal{A}} e = 0$$

so that multiplication  $P \widehat{\otimes}_{\mathcal{A}} Q \rightarrow \mathcal{B}$  is injective. It follows that  $P \widehat{\otimes}_{\mathcal{A}} Q \cong \mathcal{B}$ .

For the reversed tensor product first note that

$$(j, g, \mu) = (j, g, \lambda)(i, p_{\lambda i}^{-1}, \mu)$$

for all  $j \in I, g \in G, \mu \in \Lambda$ , so that the multiplication  $Q \widehat{\otimes}_{\mathcal{B}} P \rightarrow \mathcal{A}$  is surjective.

Identifying  $Q \widehat{\otimes} P$  with  $\ell^1(Se \times eS)$  a generic element in  $Q \widehat{\otimes} P$  has the form

$$\sum_{j, g, h, \mu} \alpha_{jgh\mu}(j, g, \lambda) \otimes (i, h, \mu).$$

Assume that

$$\sum_{j, g, h, \mu} \alpha_{jgh\mu}(j, g, \lambda)(i, h, \mu) = 0.$$

This means that

$$\sum_{\substack{g, h \\ gp_{\lambda i}h = \gamma}} \alpha_{jgh\mu} = 0$$

for each  $j \in I, \gamma \in G, \mu \in \Lambda$ . Now

$$\begin{aligned} (j, g, \lambda) \otimes_{\mathcal{B}} (i, h, \mu) &= (j, g, \lambda) \otimes_{\mathcal{B}} (i, h, \lambda)(i, p_{\lambda i}^{-1}, \mu) \\ &= (j, g, \lambda)(i, h, \lambda) \otimes_{\mathcal{B}} (i, p_{\lambda i}^{-1}, \mu) \\ &= (j, gp_{\lambda i}h, \lambda) \otimes_{\mathcal{B}} (i, p_{\lambda i}^{-1}, \mu), \end{aligned}$$

so

$$\sum_{j, g, h, \mu} \alpha_{jgh\mu}(j, g, \lambda) \otimes_{\mathcal{B}} (i, h, \mu) = \sum_{\gamma, j, \mu} \left[ \sum_{\substack{g, h \\ gp_{\lambda i}h = \gamma}} \alpha_{jgh\mu} \right] (j, \gamma, \lambda) \otimes_{\mathcal{B}} (i, p_{\lambda i}^{-1}, \mu) = 0.$$

It follows that multiplication  $Q \widehat{\otimes}_{\mathcal{B}} P \rightarrow \mathcal{A}$  is injective so that  $Q \widehat{\otimes}_{\mathcal{B}} P \cong \mathcal{A}$ . All together,  $\mathcal{A}$  and  $\mathcal{B}$  are Morita equivalent.  $\square$

We want to establish Morita invariance of Hochschild homology. We have already noted that  $P$  is strictly projective in  $\mathbf{mod}\text{-}\mathcal{A}$ . We now prove

**Theorem 2.13.** *The module  $P = e\mathcal{A}$  is strictly projective in  $\mathcal{B}\text{-mod}$ .*

*Proof.* Consider the direct sum decomposition in  $\mathcal{B}\text{-mod}$   $P = \bigoplus_{\mu} \ell^1({}_i S_{\mu})$ . Let

$$\begin{array}{ccccc} & & \ell^1({}_i S_{\mu}) & & \\ & \swarrow \text{dashed} & \downarrow \phi_{\mu} & & \\ Y & \xrightarrow{q} & Z & \longrightarrow & 0 \end{array}$$

be a lifting problem, where  $\phi_{\mu}$  is the restriction of  $\phi : P \rightarrow Z$ . From the open mapping theorem for  $q$ , there exists  $y_{\mu}$  with

$$\|y_{\mu}\| \leq C \text{ and } q(y_{\mu}) = \phi_{\mu}((i, p_{\lambda i}^{-1}, \mu)),$$

for some constant  $C$ . Define  $\tilde{\phi}_{\mu} : \ell^1({}_i S_{\mu}) \rightarrow Y$  by

$$\tilde{\phi}_{\mu}((i, g, \mu)) = (i, g, \lambda)y_{\mu}.$$

Then  $\tilde{\phi}_{\mu} \in \mathcal{B}\text{-mod}$  and  $\|\tilde{\phi}_{\mu}\| \leq C$ . Since

$$\begin{aligned} q(\tilde{\phi}_{\mu}((i, g, \mu))) &= q((i, g, \lambda)y_{\mu}) \\ &= (i, g, \lambda)q(y_{\mu}) \\ &= (i, g, \lambda)\phi_{\mu}((i, p_{\lambda i}^{-1}, \mu)) \\ &= \phi_{\mu}((i, g, \lambda)(i, p_{\lambda i}^{-1}, \mu)) \\ &= \phi_{\mu}((i, g, \mu)) \end{aligned}$$

we have solved the lifting problem. Proceeding as in 2.10 we conclude that  $P$  as a direct sum of strictly projective modules is strictly projective in  $\mathcal{B}\text{-mod}$ .  $\square$

### 3. APPLICATIONS TO HOMOLOGICAL PROPERTIES

With  $P, Q, \mathcal{A} = \mathcal{A}(S)$ , and  $\mathcal{B} = e\mathcal{A}(S)e$  as in the previous section we have functors

$$\mathcal{A}(S)\text{-mod-}\mathcal{A}(S) \rightarrow \mathcal{B}\text{-mod}\mathcal{B}: X \mapsto P \hat{\otimes}_{\mathcal{A}} X \hat{\otimes}_{\mathcal{A}} Q,$$

$$\mathcal{B}\text{-mod-}\mathcal{B} \rightarrow \mathcal{A}(S)\text{-mod-}\mathcal{A}(S): Y \mapsto Q \hat{\otimes}_{\mathcal{B}} Y \hat{\otimes}_{\mathcal{B}} Q.$$

Replacing  $\mathcal{B}$  by the isomorphic  $\ell^1(G)$  we get functors

$$\Phi : \mathcal{A}(S)\text{-mod-}\mathcal{A}(S) \rightarrow \ell^1(G)\text{-mod-}\ell^1(G),$$

$$\Gamma : \ell^1(G)\text{-mod-}\ell^1(G) \rightarrow \mathcal{A}(S)\text{-mod-}\mathcal{A}(S).$$

We collect our findings in

**Theorem 3.1.** *The functors  $\Phi$  and  $\Gamma$  constitute an equivalence of the full subcategories of induced bimodules over  $\mathcal{A}(S)$  and  $\ell^1(G)$  and there are natural isomorphisms of (co-)homology functors*

$$\mathcal{H}_n(\mathcal{A}(S), X) \cong \mathcal{H}_n(\ell^1(G), \Phi(X)),$$

$$\mathcal{H}^n(\mathcal{A}(S), X^*) \cong \mathcal{H}^n(\ell^1(G), \Phi(X)^*).$$

*Proof.* The equivalence follows from the natural isomorphisms

$$Q \widehat{\otimes}_{\mathcal{B}} (P \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{A}} Q) \widehat{\otimes}_{\mathcal{B}} P \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \cong X,$$

$$P \widehat{\otimes}_{\mathcal{A}} (Q \widehat{\otimes}_{\mathcal{B}} Y \widehat{\otimes}_{\mathcal{B}} P) \widehat{\otimes}_{\mathcal{A}} Q \cong \mathcal{B} \widehat{\otimes}_{\mathcal{B}} Y \widehat{\otimes}_{\mathcal{B}} \mathcal{B} \cong Y$$

for induced modules  $X \in \mathcal{A}(S)\text{-mod-}\mathcal{A}(S)$  and  $Y \in \mathcal{B}\text{-mod-}\mathcal{B}$ . As  $P$  is strictly projective in  $\mathcal{B}\text{-mod}$  and in  $\text{mod-}\mathcal{A}(S)$  the statement about (co-)homology groups follows from 2.3.  $\square$

We recall that  $S$  is a Rees semigroup with underlying group  $G$ . We note a number of consequences.

**Corollary 3.2.** *There are isomorphisms*

$$\mathcal{H}_n(\mathcal{A}(S), \mathcal{A}(S)) \cong \mathcal{H}_n(\ell^1(S), \ell^1(S)) \cong$$

$$\mathcal{H}_n(\mathcal{A}(S)^{\#}, \mathcal{A}(S)^{\#}) \cong \mathcal{H}_n(\ell^1(S)^{\#}, \ell^1(S)^{\#}) \cong$$

$$\mathcal{H}_n(\ell^1(G), \ell^1(G))$$

and

$$\mathcal{H}^n(\mathcal{A}(S), \mathcal{A}(S)^*) \cong \mathcal{H}^n(\ell^1(S), \ell^1(S)^*) \cong$$

$$\mathcal{H}^n(\mathcal{A}(S)^{\#}, (\mathcal{A}(S)^{\#})^*) \cong \mathcal{H}^n(\ell^1(S)^{\#}, (\ell^1(S)^{\#})^*) \cong$$

$$\mathcal{H}^n(\ell^1(G), \ell^1(G)^*)$$

for  $n \geq 0$ .

*Proof.* The proof for homology and cohomology are identical. Since the reduced module of  $\ell^1(S)$  is  $\mathcal{A}(S)$ , in both cases the first isomorphism follows from 2.6. The next two isomorphisms follow from H-unitality, cf. 2.11. Finally the last isomorphism is a consequence of 3.1 since  $\Phi(\mathcal{A}(S)) = \ell^1(G)$ .  $\square$

Recall that a Banach algebra  $\mathcal{A}$  is *weakly amenable* if  $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ .

**Corollary 3.3.** *The algebras  $\ell^1(S)^{\#}$ ,  $\ell^1(S)$ ,  $\mathcal{A}(S)^{\#}$ , and  $\mathcal{A}(S)$  are all weakly amenable.*

*Proof.* The Banach algebra  $\ell^1(G)$  is weakly amenable [14].  $\square$

Recall that a Banach algebra  $\mathcal{A}$  is *biprojective* if multiplication  $\Pi: \mathcal{A} \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$  has a right inverse in  $\mathcal{A}\text{-mod-}\mathcal{A}$ , and is *biflat* if the dual of multiplication  $\Pi^*: \mathcal{A}^* \rightarrow (\mathcal{A} \widehat{\otimes} \mathcal{A})^*$  has a left inverse in  $\mathcal{A}\text{-mod-}\mathcal{A}$ .

**Corollary 3.4.**  *$\mathcal{A}(S)$  is biflat if and only if  $G$  is amenable.*

*Proof.* By [16, Theorem 5.8.(i)] a Banach algebra  $\mathcal{A}$  is biflat if and only if it is self-induced and  $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$  for all induced modules  $X$ . As  $\ell^1(G)$ , being unital, is biflat if and only if it is amenable, the result follows from [13].  $\square$

The corresponding result for biprojectivity is not immediate from Morita theory as a description of biprojectivity in terms of Hochschild cohomology involves non-induced modules. But we can give a direct proof of

**Theorem 3.5.**  *$\mathcal{A}(S)$  is biprojective if and only if  $G$  is finite.*

*Proof.* Assume that  $|G| < \infty$ . Choose an idempotent  $e = (i, p_{\lambda i}^{-1}, \lambda)$  and define  $\rho: \mathcal{A}(S) \rightarrow \mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)$  by

$$\rho((j, g, \mu)) = \frac{1}{|G|} \sum_{h \in G} (j, gh p_{\lambda i}^{-1}, \lambda) \otimes (i, h^{-1}, \mu), \quad (j, g, \mu) \in I \times G \times \Lambda.$$

Then clearly  $\Pi \circ \rho = \mathbf{1}$ . One checks, as in the proof of biprojectivity of group algebras over finite groups, that

$$(j, g, \mu) \rho((j', g', \mu')) = \rho((j, g, \mu)(j', g', \mu')) = \rho((j, g, \mu))(j', g', \mu')$$

for all  $(j, g, \mu), (j', g', \mu') \in I \times G \times \Lambda$ , so that  $\rho$  is a bimodule homomorphism.

Conversely, suppose that  $\mathcal{A}(S)$  is biprojective, and let  $\mathcal{A}(S) \rightarrow \mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)$  be a splitting of multiplication. Consider its restriction  $\rho: e\mathcal{A}(S)e \rightarrow e\mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)e$ . Since  $\rho$  is a bimodule homomorphism we have

$$\rho(eae) = eae\rho(e) = \rho(e)eae, \quad a \in \mathcal{A}(S).$$

Using the decomposition

$$e\mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)e = \bigoplus_{j, \mu} \ell^1({}_i S_\mu \times {}_j S_\lambda)$$

as a direct sum of  $e\mathcal{A}(S)e$  bimodules, we can write

$$\rho(e) = \sum_{j, \mu} \tau_{\mu j}, \quad \tau_{\mu j} \in \ell^1({}_i S_\mu \times {}_j S_\lambda).$$

It follows that

$$eae\tau_{\mu j} = \tau_{\mu j}eae$$

for all  $a \in \mathcal{A}(S)$ ,  $j \in I$ ,  $\mu \in \Lambda$ .

In the remainder of the proof it will be convenient to use the multiplication on a projective tensor product of Banach algebras given by  $a \otimes b \cdot a' \otimes b' := aa' \otimes b'b$ .

For each  $j \in I$ ,  $\mu \in \Lambda$  choose  $f_{\mu j} \in {}_j S_\lambda$  and  $e_{\mu j} \in {}_i S_\mu$  so that

$$\Pi(\tau_{\mu j} \cdot f_{\mu j} \otimes e_{\mu j}) = \Pi(\tau_{\mu j}).$$

This is clearly possible: If  $p_{\mu j} = \mathbf{o}$  choose  $e_{\mu j}$  and  $f_{\mu j}$  arbitrarily. If  $p_{\mu j} \neq \mathbf{o}$  choose  $f_{\mu j} = (j, p_{\mu j}^{-1}, \lambda)$  and  $e_{\mu j} = (i, p_{\lambda i}^{-1}, \mu)$ .

Now put

$$\Delta = \sum_{j, \mu} \tau_{\mu j} \cdot f_{\mu j} \otimes e_{\mu j}.$$

Then  $\Delta \in e\mathcal{A}(S)e \widehat{\otimes} e\mathcal{A}(S)e$  and

$$\Pi(\Delta) = e$$

$$eae\Delta = \Delta eae, \quad a \in \mathcal{A}(S)$$

so that  $e\mathcal{A}(S)e$  has a diagonal and therefore is biprojective. Since  $e\mathcal{A}(S)e \cong \ell^1(G)$ , the group  $G$  must be finite.  $\square$

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